

Reg. No. :

Name :

Sixth Semester B.Sc. Degree Examination, March 2021

First Degree Programme under CBCSS

Mathematics

Core Course

MM 1643 – COMPLEX ANALYSIS – II

(2015 – 2017 Admission)

Time : 3 Hours

Max. Marks : 80

SECTION – A

All the first **ten** questions are compulsory. **Each** question carries **1** mark.

1. Find the power series representation of $f(z) = \frac{1}{z}$ in powers $1-z$.
2. Find the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$.
3. What are the isolated singularities of $\frac{1}{\cos \frac{\pi}{2}}$.
4. Define removable singularity.
5. What type of isolated singularity for $f(z) = \sin\left(\frac{1}{z}\right)$ has at $z = 0$.
6. Find the residue of $f(z) = \frac{1+z}{z-1}$ at $z=1$.

7. Find the pole (poles) of $f(z) = \frac{1}{z(e^z - 1)}$ and find their order.
8. Define Cauchy principal value of an improper integral.
9. Find the set of all singular points of the function $f(z) = \operatorname{Re}(z)$.
10. Evaluate $\int_C \frac{1}{(z-1)^2} dz$, where C is the circle $|z-1|=1$ taken in counter clockwise direction.

(10 × 1 = 10 Marks)

SECTION – B

Answer **any eight** questions from this section. **Each** question carries 2 marks.

11. Evaluate $\int_C \frac{z}{(9-z^2)(z+i)} dz$, where C is the circle $|z|=2$ taken in counter clockwise direction.
12. Find a power series representing the function $f(z) = \frac{1}{z^2}$ near $z=3$. Also find the region of Convergence.
13. Evaluate $\int_C \frac{e^z}{z^2 + 2z + 1} dz$, where C is the circle $|z+1|=1$ oriented in counter clockwise direction.
14. If $f(z)$ is an analytic function so that $f(z) = 3z+1$ for all z in the unit circle $C: |z|=1$. Show that $|f^5(0)| \leq 480$.
15. Find $\int_C \frac{1}{z^2 + 5} dz$, where C is the circle $|z-i|=3$ oriented positively.
16. Define residue of a complex function at an isolated singular point.
17. Find the residues of $\cot z$ at its poles.
18. State Jordan's lemma.
19. Evaluate $\sum_{n=0}^n \binom{n}{k}^2$.

20. State Cauchy's residue theorem.
21. Using Cauchy's residue theorem evaluate $\int_C \frac{e^z}{z^2} dz$, where C is the circle $|z|=3$ oriented positively.
22. Find power series representation of $\cosh z$ in powers of z .

(8 × 2 = 16 Marks)

SECTION – C

Answer any **six** questions from this section. **Each** question carries **4** marks.

23. State and prove Cauchy's integral formula.
24. Prove that for any real number a , $\int_0^\pi e^{a \cos \theta} \cos(a \sin \theta) d\theta = \pi$.
25. Using Cauchy's residue theorem evaluate $\int_C \frac{z^5}{1-z^3} dz$ where C is the circle $|z|=2$ oriented in counter clockwise direction.
26. Prove that an isolated singular point z_0 of a function f is a pole of order m if and only if $f(z)$ can be written in the form $f(z) = \frac{\phi(z)}{(z-z_0)^m}$ where $\phi(z)$ is analytic and nonzero at z_0 .
27. Find the residue of $f(z) = \frac{z dz}{(9-z^2)(z+1)}$ at its poles and hence find $\int_C f(z) dz$ where C is the positively oriented circle $|z|=2$.
28. Evaluate the improper integral $\int_0^\infty \frac{dx}{(x^2+1)^2}$.
29. Show that $\sum_{n=0}^\infty \binom{2n}{n} \frac{1}{5^n} = \sqrt{5}$.
30. Evaluate $\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta}$.
31. Show that $\sum_{n=1}^\infty \frac{(-1)^n}{n^2+1} = \frac{-1}{2} + \frac{\pi e^\pi}{e^{2\pi}-1}$.

(6 × 4 = 24 Marks)

SECTION – D

Answer any **two** questions from this section. **Each** question carries **15** marks.

32. (a) If f is an analytic at a given point, then show that the derivative of all orders of f are analytic there.
- (b) Find $g(2)$ if $g(z_0) = \int_C \frac{2s^2 - s - 2}{s - z_0} ds$, where C is the positively oriented circle $|z|=3$ and $|z_0| \neq 3$.
33. (a) If C is the positively oriented circle $|z|=1$ show that $\int_C z^2 \sin \frac{1}{z} dz = -\frac{\pi i}{3}$.
- (b) Prove that, a point z_0 at which the function $f(z)$ is analytic, is a zero of $f(z)$ of order m if and only if $f(z)$ can be written in the form $f(z) = (z - z_0)^m g(z)$ where $g(z)$ is analytic and nonzero at z_0 .
- (c) If p and q are two complex functions, which are analytic at a point z_0 so that $p(z_0) \neq 0$ and q has a zero of order m at z_0 . Show that $\frac{p(z)}{q(z)}$ has a pole of order m at z_0 .
34. (a) State and prove Cauchy's residue theorem.
- (b) Evaluate $\int_0^\infty \frac{dx}{x^4 + 1}$.
- (c) Evaluate $\int_0^\infty \frac{\cos ax}{(x^2 + b^2)^2} dx, (a > b > 0)$.
35. (a) Prove that if f is complex function having only a finite number of poles $\{z_k\}$, then
$$\sum_{n=-\infty, n \neq z_k}^{\infty} (-1)^n f(n) = -\sum_k \text{Res}(f(z) \pi \cot \pi z, z_k).$$
- (b) Evaluate $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$.

(2 × 15 = 30 Marks)