

Reg. No. :

Name :

Sixth Semester B.Sc. Degree Examination, April 2023

First Degree Programme under CBCSS

Mathematics

Core Course IX

MM 1641 : REAL ANALYSIS – II

(2018 Admission Onwards)

Time : 3 Hours

Max. Marks : 80

SECTION – A

All the first ten questions are compulsory. They carry 1 mark each.

1. Evaluate $\lim_{x \rightarrow 7/4} \frac{|x-2|}{x-2}$
2. State true or false: Every uniformly continuous function is continuous.
3. Determine the points of discontinuity of the greatest integer function.
4. State the mean value theorem.
5. Define a uniformly continuous function.
6. Define a differentiable function at a point.

7. Give an example of a real valued function which is discontinuous at every point of $\mathbb{R}$.
8. Define upper integral of a function f .
9. When do you say that a bounded real function f is integrable on $[a, b]$?
10. State true or false: If $|f|$ is integrable on $[a, b]$ then f is also integrable on $[a, b]$.

(10 × 1 = 10 Marks)

SECTION – B

Answer any **eight** questions. Each question carries 2 marks.

11. Evaluate $\lim_{x \rightarrow 0} \frac{x^2}{|x|}$

12. Prove that the Dirichlet's function f defined on $\mathbb{R}$ by $f(x) = \begin{cases} 1 & \text{if } x \text{ is irrational} \\ -1 & \text{if } x \text{ is rational} \end{cases}$ is discontinuous at every point.

13. If $f: A \rightarrow \mathbb{R}$ and $g: A \rightarrow \mathbb{R}$ are continuous at a point $c \in A$, show that $f(x) + g(x)$ is also continuous at c .

14. Is the function $f(x) = \frac{1}{x}$ uniformly continuous on $(0, 1]$? Justify.

15. Prove that $\{f(x_n)\}$ is a Cauchy sequence for every Cauchy sequence $\{x_n\}$ in $\mathbb{R}$ where f is a uniformly continuous function.

16. If f is differentiable in (a, b) and $f'(x) \leq 0$ for all $x \in (a, b)$, show that f is monotonically decreasing.

17. Show by an example that a bounded function in $[a, b]$ need not be continuous in $[a, b]$.

18. If $f: A \rightarrow \mathbb{R}$ is differentiable at a point $c \in A$, then f is continuous at c as well.

19. Find the value of δ for the function $f(x) = x^2 + 4x + 3$ to be uniformly continuous in the interval $[-1, 1]$, given $\varepsilon = \frac{1}{10}$.
20. Check whether the following function is integrable over $[0, 1]$; $f(x) = 1$ if $x \in [0, 1]$ and x is rational and $f(x) = 0$ if $x \in [0, 1]$ and x is irrational.
21. Show that $\int_a^b f dx \geq \int_a^{\bar{b}} f dx$.
22. Show that if f and g are bounded and integrable on $[a, b]$, such that $f \geq g$, then $\int_a^b f dx \geq \int_a^b g dx$.

SECTION - C

(8 × 2 = 16 Marks)

Answer any **six** questions. Each question carries 4 marks.

23. Test the continuity of the function at $x = 0$ $f(x) = \begin{cases} x \sin \frac{1}{x}, & \text{for } x \neq 0 \\ 0, & \text{for } x = 0 \end{cases}$
24. Explain Lipschitz functions with the geometrical interpretation.
25. Show that a uniformly continuous function preserves Cauchy sequences.
26. Suppose f is a real differentiable function on $[a, b]$ and suppose $f'(a) < \lambda < f'(b)$. Prove that there is a point $x \in (a, b)$ such that $f'(x) = \lambda$.
27. State and prove chain rule of differentiation.
28. State and prove Darboux's theorem.
29. Prove that, if f is monotonic in $[a, b]$, it is integrable in $[a, b]$.

30. If f and g are integrable in $[a, b]$ then show that fg is also integrable in $[a, b]$.

31. Show that the Dirichlet's function $g(x) = \begin{cases} 1 & \text{for } x \text{ rational} \\ 0 & \text{for } x \text{ irrational} \end{cases}$ is not integrable.

SECTION - D

(6 × 4 = 24 Marks)

Answer any **two** questions. Each question carries **15** marks.

32. Let $f: A \rightarrow \mathbb{R}$ be continuous on A . If $K \subseteq A$ is compact, then prove that $f(K)$ is compact as well.

33. State and prove Intermediate value theorem. Is the converse true? Justify your answer.

34. Prove that a bounded function f is integrable on $[a, b]$ if and only if for every $\varepsilon > 0$ there exists a partition P such that $U(P, f) - L(P, f) < \varepsilon$.

35. If f is bounded and integrable on $[a, b]$ and k is a number such that $|f(x)| \leq k$ for all $x \in [a, b]$. Prove that $\left| \int_a^b f(x) dx \right| \leq k(b-a)$.

(2 × 15 = 30 Marks)