

Reg. No. :

Name :

Sixth Semester B.Sc. Degree Examination, April 2023

First Degree Programme under CBCSS

Mathematics

Core Course XI

MM 1643 : ABSTRACT ALGEBRA-RING THEORY

(2018 Admission Onwards)

Time : 3 Hours

Max. Marks : 80

SECTION – A

Answer all questions. Each carries 1 mark.

1. Give an example of commutative ring with zero divisors.
2. Which are units of Z_5 ?
3. What is the characteristic of nZ ?
4. Find the number of elements in the factor ring $\frac{2Z}{8Z}$.
5. State factor theorem.
6. Is the ring $2Z$ isomorphic to the ring $4Z$?
7. Define a primitive polynomial.

8. Define a unique factorisation domain.
9. Find the norm of $1 + \sqrt{-5}$.
10. State whether true or false: Every Euclidean domain is a UFD.

(10 × 1 = 10 Marks)

SECTION – B

Answer any **eight** questions. Each carries **2** marks.

11. Prove that $a(-b) = (-a)b = -(ab)$ in a ring R .
12. Show that if a and b are idempotents in a commutative ring, then ab is also idempotent.
13. Prove that the ideal $\langle x^2 + 1 \rangle$ is not prime in $\mathbb{Z}_2[x]$.
14. Define a principal ideal domain. Give an example.
15. Find the kernel of the ring homomorphism from $R[x]$ to R defined by $f(x) \rightarrow f(1)$.
16. Is the field of real numbers is ring isomorphic to the field of complex numbers? Justify your answer.
17. Show that the polynomial $2x + 1$ in $\mathbb{Z}_4[x]$ has a multiplicative inverse in $\mathbb{Z}_4[x]$.
18. Construct a field of 9 elements.
19. Prove that every subring of $\mathbb{Z}$ is of the form $n\mathbb{Z}$ for some $n \in \mathbb{Z}$.
20. Define associates and irreducibles in an integral domain.
21. Suppose that a and b belong to an integral domain, $b \neq 0$ and a is not a unit. Show that $\langle ab \rangle$ is a proper subset of $\langle b \rangle$.
22. Let $d < -1$ be an integer, that is not divisible by the square of a prime. Prove that the only units of $\mathbb{Z}(\sqrt{d})$ are ± 1 .

(8 × 2 = 16 Marks)

SECTION – C

Answer any **six** questions. Each question carries **4** marks.

23. Prove that the set $\left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \mid a, b \in \mathbb{Z} \right\}$ is a subring of the ring of all 2×2 matrices over $\mathbb{Z}$.
24. Find all solutions of the equation $x^2 - 5x + 6 = 0$ in $\mathbb{Z}_{12}$.
25. Show that an intersection of subfields of a field F is again a sub field of F .
26. Let ϕ be a ring homomorphism from a ring R to a ring S . If B is an ideal in S , then prove that $\phi^{-1}(B) = \{r \in R \mid \phi(r) \in B\}$ is an ideal of R .
27. Let R be a ring with unity 1 . Then show that the mapping $\phi: \mathbb{Z} \rightarrow R$ by $\phi(m) = m \cdot 1$ is a ring homomorphism.
28. Show that if D is an integral domain, then $D[x]$ is also an integral domain.
29. Show that $1 + \sqrt{-3}$ is irreducible in $\mathbb{Z}[\sqrt{-3}]$.
30. Show that the ring of Gaussian integers $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$ is a Euclidean domain.
31. In $\mathbb{Z}[\sqrt{-5}]$, show that 21 doesn't factor uniquely as a product of irreducibles.

SECTION – D

(6 × 4 = 24 Marks)

Answer any **two** questions. Each carries **15** marks.

32. (a) Prove that a finite integral domain is a field.
- (b) Prove that the characteristic of an integral domain is zero or prime.

33. Let D be an integral domain. Then prove that there exist a field F that contains a subring isomorphic to D .
34. (a) State and prove Gauss's lemma.
- (b) Let $f(x) \in \mathbb{Z}[x]$. Prove that if $f(x)$ is reducible over $\mathbb{Q}$, then it is reducible over $\mathbb{Z}$.
35. Prove that in a PID, an element is irreducible if and only if it is prime.

(2 × 15 = 30 Marks)