

Reg. No. :

Name :

Fifth Semester B.Sc. Degree Examination, December 2023

First Degree Programme under CBCSS

Mathematics

Core Course VII

MM 1543 : ABSTRACT ALGEBRA - GROUP THEORY

(2018 Admission Onwards)

Time : 3 Hours

Max. Marks : 80

SECTION I

(Answer all questions. They carry 1 mark each)

1. When do you say that a binary operation is commutative.
2. Determine whether the binary operation 'subtraction of integers' is associative.
3. When we say that two algebraic structures are isomorphic.
4. Is a group of order 3 cyclic?
5. Compute $(1,5,3)(2,4)$.
6. State why a group of order 12 must have an element of order 2.
7. Define a transposition.
8. State true or false : Every group of prime order is abelian.
9. State true or false: Every permutation is a cycle.
10. Find all cosets of the subgroup $\langle 2 \rangle$ of $\mathbb{Z}_{12}$.

(10 × 1 = 10 Marks)

P.T.O.

SECTION II

Answer **any eight** questions. These questions carry **2** marks each.

11. Prove that in a group, $(a^{-1})^{-1} = a$ for all a .
12. Show that $(2\mathbb{Z}, +)$ is isomorphic to $(\mathbb{Z}, +)$.
13. Why is the set of odd integers not a group under addition?
14. Show that the left and right cancellation law holds in a group.
15. Find all the generators of the cyclic group of complex numbers $G = \{1, -1, i, -i\}$. Find a subgroup of G .
16. For an abelian group G , prove that $(ab)^{-1} = a^{-1}b^{-1}$ for all a and b in G .
17. If H and K are subgroups of a group G , prove that $H \cap K$ is also a group.
18. Let H and K be subgroups of a group G . If $|H| = 10$ and $|K| = 21$, prove that $H \cap K = \{e\}$, where e is the identity of G .
19. Compute $\beta\alpha$ where $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 3 & 5 & 1 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 1 & 2 & 3 \end{pmatrix}$.
20. Let H be a subgroup of G , and let a and b belong to G . Prove that $(ab)H = a(bH)$ and $H(ab) = (Ha)b$.
21. Find the orbit containing 1 of the permutation $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 8 & 6 & 7 & 4 & 1 & 5 & 2 \end{pmatrix}$.
22. Find the index of $\langle \alpha \rangle$ in the group S_3 .

(8 × 2 = 16 Marks)

SECTION III

Answer **any six** questions. These questions carry **4** marks each .

23. Prove that in a group, $(ab)^2 = a^2b^2$ if and only if $ab = ba$.
24. Prove that the identity element and inverse of each element in a group are unique.
25. Prove that every subgroup of a cyclic group is cyclic.
26. Prove that commutative property is invariant under homomorphism.
27. Let G be an Abelian group and H and K be subgroups of G . Show that $HK = \{hk \mid h \in H, k \in K\}$ is a subgroup of G
28. Let H be a normal subgroup of a group G and K be any subgroup of G . Then $HK = \{hk \mid h \in H, k \in K\}$ is a subgroup of G .
29. Let $3\mathbb{Z} = \{0, \pm 3, \pm 6, \pm 9, \dots\}$. Exhibit the left cosets and right cosets of $3\mathbb{Z}$ in $\mathbb{Z}$.
30. Let G be the real numbers under addition and let $\bar{G}$ be the positive real numbers under multiplication. Prove that G and G are isomorphic under the mapping $f(x) = 2^x$.
31. State the first Isomorphism theorem.

(6 × 4 = 24 Marks)

SECTION IV

Answer **any two** questions. These question carry **15** marks each.

32. (a) Prove that a non-empty subset H of a group G is a subgroup of G if and only if
 - (i) H is closed under the binary operation of G
 - (ii) The identity element e of G is in H ,
 - (iii) For all $a \in H$ it is true that $a^{-1} \in H$.
- (b) Let G be a group and $a \in G$. Prove that $H = \{a^n \mid a^n \in \mathbb{Z}\}$ is a subgroup $H = \{a^n \mid n \in \mathbb{Z}\}$ is a subgroup of G and is the smallest subgroup of G that contains a .

33. Let f be a homomorphism from a group G to a group $\bar{G}$ and let H be a subgroup of G , then prove that

- (a) $f(H) = \{f(h) \mid h \in H\}$ is a subgroup of $\bar{G}$.
- (b) If H is cyclic, then $f(H)$ is cyclic
- (c) If H is Abelian, then $f(H)$ is Abelian.
- (d) If H is normal in G , then $f(H)$ is normal in $f(G)$
- (e) If f is onto and $\text{Ker } f = \{e\}$, then f is an isomorphism from G to $\bar{G}$.

34. State and prove Lagrange's theorem. Is the converse true? Justify your answer.

35. (a) Prove that every permutation σ of a finite set is a product of disjoint cycles

(b) Compute the product of cycles $(1, 4, 5) (7, 8) (2, 5, 7)$ that is a permutation of $\{1, 2, 3, 4, 5, 6, 7, 8\}$

(c) Express the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 6 & 4 & 1 & 8 & 2 & 5 & 7 \end{pmatrix}$ as a product of disjoint cycles and then as a product of transpositions.

(2 × 15 = 30 Marks)