

Reg. No. :

Name :

Sixth Semester B.Sc. Degree Examination, March 2020

First Degree Programme Under CBCSS

Mathematics

Core Course IX

MM 1641 : REAL ANALYSIS – II

(2014 admission onwards)

Time : 3 Hours

Max. Marks : 80

SECTION – I

All the first 10 questions are compulsory. Each carries 1 mark :

1. Is the function $f(x) = \frac{1}{x}$ continuous on R ?
2. Prove that the sum of two real valued continuous functions is continuous.
3. If $f : A \rightarrow R$, $A \subseteq R$ is continuous on A , then prove that $|f|$ is continuous on A .
4. Write the sequential criterion for continuity.
5. Is the derivative of the function $f(x) = |x|$, $x \in R$ exists at 0? Justify.
6. Differentiate and simplify $g(x) = \sqrt{5 - 2x + x^2}$.
7. State Rolle's Theorem.

8. Evaluate $\lim_{x \rightarrow 0} \left(\frac{1 - \cos x}{x^2} \right)$.
9. Define Riemann integrable functions on $[a, b]$.
10. Prove that every constant function on $[a, b]$ is Riemann integrable.

SECTION – II

Answer **any eight** questions from this section. Each question carries 2 marks.

11. Let $f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$. Show that f is not continuous at any point of R .
12. Let $K > 0$ and let $f: R \rightarrow R$ satisfy the condition $|f(x) - f(y)| \leq K|x - y|$ for all $x, y \in R$. Show that f is continuous at every point $c \in R$.
13. Give an example of a function $f: [0, 1] \rightarrow R$ that is discontinuous at every point of $[0, 1]$ but $|f|$ is continuous on $[0, 1]$.
14. Find a closed bounded interval in which the equation $f(x) = xe^x - 2 = 0$ has a root.
15. Give an example to show that the continuous image of an open interval need not be an open interval.
16. Define monotone functions. Give an example of a monotone function which is not continuous.
17. Prove that all differentiable functions are continuous.
18. Let $I = \left(0, \frac{\pi}{2}\right)$. Then find $\lim_{x \rightarrow 0^+} \left(\frac{1}{x} - \frac{1}{\sin x} \right)$.
19. State Taylor's Theorem. Using Taylor's theorem approximate the function $\sqrt[3]{1+x}$, $x > -1$ with $n = 2$.

20. If $I = [0, 4]$, calculate the norm $\|\mathcal{P}\|$ of the partition $\mathcal{P} = (0, 1, 2, 4)$. Also find the Riemann sum $S(f; \mathcal{P})$ where $f(x) = x^2$ and tags at the right endpoints of the subintervals.
21. Prove that if $f \in \mathcal{R} [a, b]$ then the value of the integral is unique.
22. Prove that if $f, g \in \mathcal{R} [a, b]$ then the sum $f + g \in \mathcal{R} [a, b]$.

SECTION – III

Answer **any six** questions from this section. Each question carries 4 marks.

23. Let $I = [a, b]$ be a closed bounded interval and let $f: I \rightarrow \mathbb{R}$ be continuous on I . Then prove that f is bounded on I .
24. Let I be an interval and let $f: I \rightarrow \mathbb{R}$ be continuous on I . If $a, b \in I$ and $k \in \mathbb{R}$ satisfies $f(a) < k < f(b)$, then prove that there exists a point $c \in I$ between a and b such that $f(c) = k$.
25. Let I be an interval and let $f: I \rightarrow \mathbb{R}$ be continuous on I . Prove that the set $f(I)$ is an interval.
26. Let $f: I \rightarrow \mathbb{R}$ be increasing on I . Suppose that $c \in I$ is not an end point of I . Prove that $\lim_{x \rightarrow c^-} f(x) = \sup \{f(x) : x \in I, x < c\}$.
27. Define derivative of a real valued function f at c . State and prove product rule of differentiation.
28. Define relative extremum of a real valued function. Prove that if the function $f: I \rightarrow \mathbb{R}$ has a relative extremum at an interior point c of I and the derivative $f'(c)$ exists, then $f'(c) = 0$.
29. Prove that if $f \in \mathcal{R} [a, b]$ then f is bounded on $[a, b]$.
30. Define a step function. Prove that if $\varphi: [a, b] \rightarrow \mathbb{R}$ is a step function then $\varphi \in \mathcal{R} [a, b]$.
31. Define antiderivative of a function $f: [a, b] \rightarrow \mathbb{R}$. State fundamental theorem of Calculus (First Form) and apply the theorem for the function $f(x) = x$ for all $x \in [a, b]$.

SECTION – IV

Answer **any two** questions from this section. Each question carries **15** marks.

32. (a) Define absolute minimum and absolute maximum of a real valued function f . Prove that if f is continuous on a closed bounded interval I , then f has an absolute maximum and an absolute minimum on I .
- (b) Prove that if $f: I \rightarrow R$ be continuous on $I = [a, b]$ then $f(I)$ is a closed bounded interval.
33. (a) Define jump of an increasing function $f: I \rightarrow R$ at an interior point $c \in I$. Prove that the set of point $D \subseteq I$ at which f is discontinuous is a countable set.
- (b) State and prove Chain Rule of differentiation.
34. (a) State and prove mean value theorem.
- (b) State and prove Cauchy Criterion for Riemann integrability.
35. (a) Prove that if $f: [a, b] \rightarrow R$ is continuous on $[a, b]$, then $f \in \mathcal{R} [a, b]$.
- (b) Prove that if $f: [a, b] \rightarrow R$ is monotone on $[a, b]$, then $f \in \mathcal{R} [a, b]$.
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