

SECTION – D

Answer any two questions. Each question carries 15 marks.

32. (a) If $f(z)$ is analytic in the disk $|z - z_0| < R$, then prove that the Taylor series for $f(z)$ around z_0 converges to $f(z)$ for all z in the disk $|z - z_0| < R$. 8

- (b) Expand $f(z) = \frac{1}{(z-1)^2(z-3)}$ in a Laurent's series valid for

(i) $0 < |z-1| < 2$

(ii) $0 < |z-3| < 2$ 7

33. (a) Establish Cauchy's residue theorem. 8

- (b) Using Residues, find the Cauchy principle value of $\int_{-\infty}^{\infty} \frac{dx}{(x^2 - 3x + 2)(x^2 + 1)}$. 7

34. (a) Find the image of $|z - 2i| = 2$ under the transformation $w = \frac{1}{z}$. 7

- (b) Show that the transformation $w = \frac{i(i-z)}{1+z}$ maps the circle $|z| < 1$ into the real axis of the w -plane and interior of the circle into upper half of the w -plane. 8

35. (a) State Reimann mapping theorem. 3

- (b) Construct the Mobius transformation that takes the points $-1, i, 1$ onto $0, i, \infty$. 5

- (c) Discuss the image of the circle $|z-2|=1$ under the transformation $w = \frac{z-4}{z-3}$. 7

(2 × 15 = 30 Marks)

(Pages : 4)

T – 1606

Reg. No. :

Name :

Sixth Semester B.Sc. Degree Examination, April 2024

First Degree Programme under CBCSS

Mathematics

Core Course X

MM 1642 : COMPLEX ANALYSIS – II

(2018–2020 Admission)

Time : 3 Hours

Max. Marks : 80

SECTION – A

All the first ten questions are compulsory. They carry 1 mark each.

1. State Ratio test for convergence of the complex series $\sum_{j=0}^{\infty} c_j$.
2. Suppose $f(z)$ is analytic inside a circle C with centre at z_0 and $f(z) = \sum_{k=0}^{\infty} a_k(z - z_0)^k$. What is the value of a_k ?
3. Write the power series representation of $f(z) = \frac{1}{z-1}$ in powers of z .
4. Classify the singularity at $z = 0$ of the function $f(z) = \frac{\sinh z}{z^5}$.
5. Define residue of a function f at z_0 , where z_0 is an isolated singularity of $f(z)$.
6. Find Cauchy principal value of the improper integral $\int_{-\infty}^{\infty} x dx$.

7. Show that $f(z) = z^2 + 2$ has a pole of order 2 at ∞ .
8. Define the cross ratio of the four points z, z_1, z_2, z_3 .
9. State Jordan's lemma.
10. Define Mobius transformation.

SECTION – B

(10 × 1 = 10 Marks)

Answer any **eight** questions. Each question carries **2** marks.

11. Using comparison test, show that the series $\sum_{j=1}^{\infty} \frac{1}{j(j+1)}$ converges.
12. Find the Maclaurin series of $\sinh z$.
13. Find the circle of convergence of the power series $\sum_{k=0}^{\infty} 3^k (z-1)^k$.
14. Find the radius of convergence of the power series $\sum_{k=0}^{\infty} \frac{2^k}{k^2} z^k$.
15. Write down the principal part of the function $z \exp\left(\frac{3}{z}\right)$ at its isolated singular point and determine the nature of the singularity.
16. Determine and identify the singular points of $\frac{1}{z(z^2+1)}$.
17. Determine the order of the pole and residue of the function $\frac{\sin z}{z^4}$ at $z=0$.
18. Let $f(z) = \frac{P(z)}{Q(z)}$, where the functions $P(z)$ and $Q(z)$ are both analytic at z_0 and Q has a simple zero at z_0 while $P(z_0) \neq 0$. Prove that $\operatorname{Res}_{z \rightarrow z_0} f(z) = \frac{P(z_0)}{Q'(z_0)}$.
19. Show that $\int_C \exp\left(\frac{1}{z^2}\right) dz = 0$, where C is the positively oriented circle $|z|=1$.
20. Show that the polynomial $f(z) = z^3 - 8$ has a zero of order $m=1$ at $z_0 = 2$.

21. Find the fixed points of the Mobius transformation $f(z) = \frac{z-1}{z+1}$.
22. Find the points in the z -plane at which the mapping $f(z) = z^5 - 5z$ is not conformal.

(8 × 2 = 16 Marks)

SECTION – C

Answer any **six** questions. Each question carries **4** marks.

23. Expand $f(z) = \frac{1}{z^2}$ as a Taylor series about $z=3$. State the region of validity.
24. Find the Laurent's series expansion for the function $f(z) = \frac{1}{z(z-3)}$ valid for
 - (a) $0 < |z| < 3$
 - (b) $|z| > 3$
25. Describe the classification of isolated singular points of a function.
26. Using Cauchy's residue theorem evaluate $\int_C \tan z dz$ where C denote the positively oriented circle $|z|=2\pi$.
27. Compute P.V $\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + 9} dx$.
28. Using the method of residues, prove that $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$.
29. Show that $\int_0^{2\pi} \frac{d\theta}{1 + a \cos \theta} = \frac{2\pi}{\sqrt{1-a^2}} (-1 < a < 1)$.
30. Prove that an analytic function f is conformal at every point z_0 for which $f'(z_0) \neq 0$.
31. Find the image of the unit circle $|z|=1$ under the linear fractional transformation $f(z) = \frac{z+2}{z-1}$.

(6 × 4 = 24 Marks)