

Reg. No. :

Name :

Sixth Semester B.Sc. Degree Examination, March 2020

First Degree Programme under CBCSS

Mathematics

Core Course XI

MM 1643 : COMPLEX ANALYSIS – II

(2014 Admission onwards)

Time : 3 Hours

Max. Marks : 80

SECTION – I

All the first **ten** questions are compulsory. Each carries **1** mark.

1. Find a power series expansion for $\frac{1}{z^2}$ near $z = 3$.
2. If f is analytic in a disc $B(\alpha, r)$ and $f(z) = \sum_{k=0}^{\infty} c_k (z - \alpha)^k$, what is the value of c_k ?
3. What you meant by an isolated singular point of a function $f(z)$?
4. Write the principal part of the function $f(z) = ze^{\frac{1}{z}}$ at its isolated singular point.
5. Find the residue of the function $f(z) = \frac{1 - \cosh z}{z^3}$ at its pole.

6. Find singularity of the function $f(z) = \frac{1}{z^2(1+z)}$.
7. What is meant by an entire function?
8. What is the order of pole $z = 0$ of $\frac{1}{z(e^z - 1)}$.
9. What type of singularity the function $f(z) = e^{\frac{1}{z}}$ has at $z = 0$?
10. Show that the point $z = 0$ is a simple pole of the function $f(z) = \frac{1}{\sin z}$.

(10 × 1 = 10 Marks)

SECTION – II

Answer **any eight** questions from this section. Each question carries **2** marks.

11. If f is analytic in $D(\alpha; r)$ and $a \in D(\alpha; r)$, then prove that there exist functions F and G , analytic in D and such that $F'(z) = f(z)$, $G'(z) = \frac{f(z) - f(a)}{z - a}$.
12. Find residue of $f(z) = \frac{z+1}{z^2+9}$ at its poles.
13. Suppose that a function f is analytic throughout the finite plane except for a finite number of singular points $z_1, z_2, \dots, z_n$. Show that

$$\underbrace{\text{Res}}_{z=z_1} f(z) + \underbrace{\text{Res}}_{z=z_2} f(z) + \dots + \underbrace{\text{Res}}_{z=z_n} f(z) + \underbrace{\text{Res}}_{z=\infty} f(z) = 0$$

14. Evaluate $\int_C e^{\frac{1}{z^2}} dz$, where C is positively oriented circle $|z| = 1$.
15. Evaluate $\int_C \frac{5z-2}{z(z-1)} dz$, where C is positively oriented circle $|z| = 2$.
16. Find residue at $z = 0$ of the function $f(z) = z \cos\left(\frac{1}{z}\right)$.
17. State Cauchy's integral formula.
18. Write the principal part of the function $f(z) = \frac{\sin z}{z}$ at its poles and determine whether that point is a pole, removable singularity, or essential singularity.
19. If z_0 is a removable singular point of a function f , then prove that f is analytic and bounded in some deleted neighbourhood $0 < |z - z_0| < \varepsilon$ of z_0 .
20. State Jordan's Lemma.
21. Find $\sum_{n=1}^{\infty} \frac{1}{n^2}$
22. Evaluate $\sum_{k=0}^n \binom{n}{k}^2$.

SECTION - III

(8 × 2 = 16 Marks)

Answer **any six** question from this section. Each question carries 4 marks.

23. Suppose that a function f is analytic at z_0 and write $g(z) = \frac{f(z)}{z - z_0}$.
 - (a) If $f(z_0) \neq 0$ then prove that z_0 is a simple pole of $g(z)$ with residue $f(z_0)$
 - (b) If $f(z_0) = 0$ then prove that z_0 is a removable singular point of $g(z)$

24. Evaluate using Cauchy's Residue theorem $\int_C z^2 \sin\left(\frac{1}{z}\right) dz$ where C is the positively oriented unit circle $|z| = 1$.
25. Suppose that two functions p and q are analytic at a point z_0 and $p(z_0) \neq 0$ and q has a zero of order m at z_0 . Then prove that the quotient $p(z)/q(z)$ has a pole of order m at z_0 .
26. Let a function f be analytic at a point z_0 . Prove that $f(z)$ has a zero of order m at z_0 if and only if there is a function $g(z)$, which is analytic and nonzero at z_0 , such that $f(z) = (z - z_0)^m g(z)$.
27. Describe three types of isolated singular points.
28. Given a function f and a point z_0 , suppose that f is analytic at z_0 and $f(z_0) = 0$ but $f(z)$ is not identically equal to zero in any neighbourhood of z_0 . Then prove that $f(z) = 0$ throughout some deleted neighbourhood $0 < |z - z_0| < \varepsilon$ of z_0 .
29. Find residue of
- (a) $\cot z$ at $z - n\pi$
- (b) $\frac{\tanh z}{z^2}$ at $z = \frac{\pi i}{2}$
30. Evaluate $\int_C \frac{3z^3 + 2}{(z^2 + 9)(z - 1)} dz$ where c is the circle $|z| = 4$
31. Find $\sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{5^n}$.

(6 × 4 = 24 Marks)

Answer **any two** questions from this section. Each question carries **15** marks.

32. (a) If a function $f(z)$ is analytic every where in the finite plane except for a finite number of singular points interior to a positively oriented simple closed contour C , then prove that $\int_C f(z) dz = 2\pi i \underbrace{\text{Res}}_{z=0} \left[\frac{1}{z^2} f\left(\frac{1}{z}\right) \right]$.

(b) Evaluate $\int_C \frac{(3z+2)^2}{z(z-1)(2z+5)} dz$ where C is the circle $|z|=3$

33. (a) Let two functions p and q be analytic at a point z_0 . If $p(z_0) \neq 0$, $q(z_0) = 0$ and $q'(z_0) \neq 0$, then prove that z_0 is a simple pole of the quotient $p(z)/q(z)$ and

$$\underbrace{\text{Res}}_{z=z_0} \left(\frac{p(z)}{q(z)} \right) = \frac{p(z_0)}{q'(z_0)}$$

(b) Show that $\underbrace{\text{Res}}_{z=\pi i} \frac{z - \sinh z}{z^2 \sinh z} = \frac{i}{\pi}$

34. (a) Evaluate $\int_0^{\infty} \frac{dx}{(x^2+1)^2}$.

(b) Prove that $\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta} = \frac{2\pi}{3}$

35. (a) Use residue to find Cauchy's principal value of the integral $\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + 2x + 2} dx$

(b) Show that $\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$.

(2 × 15 = 30 Marks)

St. Xaviers College, Thumba