

Reg. No. :

Name :

Sixth Semester B.Sc. Degree Examination, March 2020

First Degree Programme under CBCSS

Mathematics

Core Course XII

MM 1644 ABSTRACT ALGEBRA II

(2014 admission onwards)

Time : 3 Hours

Max. Marks : 80

SECTION – I

All the first ten questions are compulsory. Each carries 1 marks.

1. Define a ring homomorphism.
2. What is a commutative ring?
3. What are the units in $\mathbb{Z}_3$?
4. What is the characteristic of the ring $\langle \mathbb{Q}, +, \cdot \rangle$?
5. Compute $\phi(22)$
6. Find all solutions of the congruence $8x \equiv 5 \pmod{4}$
7. Define a strictly skew field.
8. Compute $(3)(5)$ in the ring $\mathbb{Z}_6$.
9. State Fermat's little theorem.
10. How many homomorphisms are there of $\mathbb{Z}$ into $\mathbb{Z}_2$?

SECTION – II

Answer **any eight** questions from this section.

Each question carries 2 marks.

11. If H is a normal subgroup of G , then prove that $f: G \rightarrow \frac{G}{H}$ given by $f(x) = xH$ is a homomorphism with kernel H .
12. Check whether $\phi: \mathbb{Z} \rightarrow \mathbb{Z}$ defined by $\phi(x) = 2x$ for $x \in \mathbb{Z}$ is a ring homomorphism.
13. Which are the elements of the field $\mathbb{Z}_{13}$ that are their own multiplicative inverse?
14. Classify the group $\frac{(\mathbb{Z}_4 \times \mathbb{Z}_2)}{(\{0\} \times \mathbb{Z}_2)}$ according to the fundamental theorem of finitely generated abelian groups.
15. State the fundamental homomorphism theorem for rings.
16. What are zero divisors? Give example.
17. Give an example to show that a factor ring of an integral domain may have divisors of zero.
18. Show that an intersection of ideals of a ring R is again an ideal of R .
19. Find the order of the ring $M_2(\mathbb{Z}_2)$.
20. Show that $a^2 - b^2 = (a + b)(a - b)$ for all a and b in a ring if and only if R is commutative.
21. Let $\mathcal{F}$ be the additive group of continuous functions with domain $[0, 1]$ and let $\mathbb{R}$ be the additive group of real numbers. Show that the map $\sigma: \mathcal{F} \rightarrow \mathbb{R}$ defined by
$$\sigma(f) = \int_0^1 f(x) dx, f \in \mathcal{F}$$
 is a homomorphism.
22. Give an example of a ring homomorphism $\phi: R \rightarrow R^1$ where R has unity 1 and $\phi(1) \neq 0$ but $\phi(1)$ is not unity for R^1 .

SECTION – III

Answer **any six** questions from this section.

Each question carries 4 marks.

23. Let r be the natural map of $\mathbb{Z}$ into $\mathbb{Z}_n$ given by $r(m)=r$, where r is the remainder given by the division algorithm when m is divided by n . Show that r is a homomorphism.
24. Show that if G, G' and G'' are groups and $\phi: G \rightarrow G'$ and $r: G' \rightarrow G''$ are homomorphism, then the composite map $r\phi: G \rightarrow G''$ is a homomorphism.
25. Show that an intersection of normal subgroups of a group G is again a normal subgroup of G .
26. Prove that $\langle n\mathbb{Z}, +, \cdot \rangle$ is a ring.
27. Find the units in $\mathbb{Z}_{20}$.
28. Compute the remainder of 8^{103} when divided by 13.
29. Prove that cancellation laws hold in a ring R if and only if R has no divisors of zero.
30. Find all solutions of the equation $x^2 + 2x + 2 = 0$ in $\mathbb{Z}_6$.
31. Let H be a subring of the ring R . Show that multiplication of additive cosets of H is well defined by the equation $(a + H)(b + H) = ab + H$ if and only if $ah \in H$ and $hb \in H$ for all $a, b \in R$ and $h \in H$.

SECTION – IV

Answer **any two** questions from this section.

Each question carries 15 marks.

32. (a) Let $\phi: G \rightarrow G'$ be a group homomorphism and let $H = \ker(\phi)$. Let $a \in G$. Then prove that the set $\phi^{-1}[\{\phi(a)\}] = \{x \in G \mid \phi(x) = \phi(a)\}$ is the left coset aH of H , and is also the right coset Ha of H . Also prove that the two partitions of G into left cosets and into right cosets of H are the same.
- (b) Prove that a group homomorphism $\phi: G \rightarrow G'$ is one-to-one map if and only if $\ker(\phi) = \{e\}$.

33. (a) Prove that every field is an integral domain.
- (b) Prove that every finite integral domain is a field.
- (c) Prove that in the ring $\mathbb{Z}_n$, the divisors of zero are precisely those non zero elements that are not relatively prime to n .
34. (a) State and prove Euler's theorem.
- (b) Let m be a positive integer and let $a, b \in \mathbb{Z}_m$. Let d be the gcd of a and m . Then prove that the equation $ax = b$ has a solution in $\mathbb{Z}_m$ if and only if d divides b . When d divides b , then prove equation has exactly d solutions in $\mathbb{Z}_m$.
35. (a) Show that if a finite group G has exactly one subgroup H of a given order, then H is a normal subgroup of G .
- (b) Show that if H and N are subgroups of a group G and N is normal in G , then $H \cap N$ is normal in H . Show by an example that $H \cap N$ need not be normal in G .
-